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Chebyshev Approximation by Interpolating Rationals

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario,
London, Ontario N6A 5B7, Canada

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Let $[\alpha, \beta]$ be a closed finite interval and $C[\alpha, \beta]$ the space of real continuous functions on $[\alpha, \beta]$ with norm

$$\|h\| = \max\{|h(x)|: \alpha \leqslant x \leqslant \beta\}.$$

Let l and m be fixed nonnegative integers. Let $R_m^l[\alpha, \beta]$ be the set of ratios P/Q, P a real polynomial of degree l, Q, of degree m, $Q > 0$ on $[\alpha, \beta]$. Let B be a fixed element of $C[\alpha, \beta]$ not identically zero and p a fixed positive number. Let $W = \{B * [P/Q]^p: P/Q \in R_m^l[\alpha, \beta],\ P > 0 \text{ on } [\alpha, \beta]\}$. Let $f(x) = B(x) * g(x)$, $g \in C[\alpha, \beta]$, $g > 0$ on $[\alpha, \beta]$. Then our approximation problem is to find $w^* = B * [P^*/Q^*]^p \in W$ minimizing

$$\|f - B * [P/Q]^p\| = \|B(g - [P/Q]^p)\|$$

over $B * [P/Q]^p \in W$. Such an element w^* is called a best approximation to f.

The family W of approximations is a restriction of the family of approximations of Schmidt [8], who permits P to be $\geqslant 0$ on $[\alpha, \beta]$. The family W is a generalization of the family of approximations of Williams [10] who had $l = 0$.

Reasons for preferring this problem to the problem of Schmidt are as follows. First, the constraint $P > 0$ is consistent with the constraint $g > 0$. If we let $P \geqslant 0$ we should also let $g \geqslant 0$. Second, it is an open question whether P/Q can be optimal in Schmidt's problem for f, not an approximant, if P has a zero. For example, let $B = 1$ and $l = 0$. Zero can never be best to $g > 0$, as there is a constant c strictly between 0 and g. Third, as we will see, the theory for $P > 0$ is simpler and there is an algorithm for computing best approximations (if they are of maximum degree).

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DEFINITION. Let ∂ denote an exact degree. For $P \not\equiv 0$ the degree of P/Q with numerator and denominator relatively prime is $l + m + 1 - d(P/Q)$, where

$$d(P/Q) = \min\{l - \partial P, m - \partial Q\}.$$

It is known that $R_m^l[\alpha, \beta]$ is varisolvent with degrees as defined above and a degree for zero $(l + 1)$. Meinardus and Schwedt showed that $R_m^l[\alpha, \beta]$ satisfied their nonlinear Chebyshev hypotheses [7, pp. 160–161; 8, pp. 311–312]. Barrar and Loeb [1] in turn showed that the hypotheses of Meinardus and Schwedt implied varisolvence.

THEOREM. *Let U be varisolvent and v be in $C[\alpha, \beta]$; $\{u: u \in U, u > v\}$ is also varisolvent with the same degrees.*

This theorem follows easily from the definition of varisolvence. We let $v = 0$.

THEOREM. *Let U be varisolvent with elements >0, then U^p is varisolvent with degrees the same.*

This follows from Theorem 1 of Kaufman and Belford [5].

From the above it follows that $V = \{[P/Q]^p: P/Q \in R_m^l[\alpha, \beta], P > 0\}$ is verisolvent with degrees the same as for $R_m^l[\alpha, \beta]$. Alternatively we can deduce that

$$U = \{P/Q: P/Q \in R_m^l[\alpha, \beta],\ P > 0 \text{ on } [\alpha, \beta]\},$$

satisfies the nonlinear Chebyshev hypotheses of Meinardus and Schwedt, hence so does U^p. Thus the phenomenon of an optimal nonzero constant error curve discussed by Ling and Tornga [6] cannot occur [6, p. 57].

The approximation problem can be considered as an approximation of g by V with multiplicative weight B. This in turn is equivalent to an approximation of g by V with nonnegative multiplicative weight $|B|$. Approximation with respect to nonnegative weights is covered in the author's paper [4], from which we obtain

THEOREM. *$B * [P/Q]^p$ of degree n is best to $f = B * g$ if and only if $|B|(g - [P/Q]^p)$ alternates n times on $[\alpha, \beta]$. A best approximation is unique.*

An analogue of the lemma of de la Vallée–Poussin applies [3, p. 226]. The alternation result suggests use of the Remez algorithm if the best approximation is of maximum degree. The analysis of Kahan as written by the author [3] applies as modified in [4]. A version of the Remez algorithm that can be adapted to the problem of this paper is given by the author [2]. We take $w = 1/|B|$, $\phi(y) = y^p$, $\phi^{-1}(y) = y^{1/p}$.

THEOREM. *Let* $w^* = B[P^*/Q^*]^p$ *be best in* W, *then* w^* *is best in Schmidt's problem.*

Proof. Suppose not, then there is P^s/Q^s such that $P^s \geqslant 0$, $Q^s > 0$, and

$$\|B * (g - [P^s/Q^s]^p)\| < \|B * (g - [P^*/Q^*]^p)\|.$$

Consider now the approximation $\bar{w} = B * [[P^* + P^s]/[Q^* + Q^s]]^p$. It is between w^* and $B[P^s/Q^s]^p$ by betweeness arguments of the author [11, p. 152], hence $\|Bg - \bar{w}\| \leqslant \|Bg - w^*\|$. But w^* is uniquely best in W and $\bar{w} \in W$.

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